

Protection of Complex Networks against SIR Spreading Processes

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Abstract—This paper introduces a theoretical framework for the analysis and control of the stochastic susceptible-infected-removed (SIR) spreading process over a network of heterogeneous agents. In our analysis, we analyze the exact networked Markov process describing the SIR model, without resorting to mean-field approximations, and introduce a convex optimization framework to find an efficient allocation of resources to contain the expected number of accumulated infections over time. Numerical simulations are presented to illustrate the effectiveness of the obtained results.

I. INTRODUCTION

The analysis of contagion processes in complex networks is one of the central problems in network science and engineering, with applications in a wide range of scenarios, such as epidemiology [1], public health [2], and cyber-physical systems [3]. During the last decade, we have witnessed a tremendous advance in this problem, including the relationship between epidemic thresholds and network eigenvalues [4], the connection between curing policies and the cut-width of the graph [5], [6], the use of optimization tools to contain epidemic outbreaks [7]–[9], as well as new modeling frameworks for analysis of spreading processes over multilayer [10]–[12], time-varying [13], and adaptive networks [14], [15].

Designing strategies to contain epidemics outbreaks in networks is of great relevance in public health. In this context, the following question is of particular interest: given a contact network and resources that provide partial protection, how should one distribute these resources throughout the networks in a cost-optimal manner to contain the spread? This question has been addressed in several papers by the control community (see [1] and references therein). Most existing results are based on the analysis of the susceptible-infected-susceptible (SIS) spreading model, in which nodes in the network can only be in two states: infected or healthy. However, in many practical settings, nodes can also be immune to the disease due to, for example, a previous exposition to the infectant. The addition of this third state has a nontrivial effect on the dynamics of the spread, which is commonly modeled using the susceptible-infected-removed (SIR) epidemic model [1], in which a node recovers from the infection with acquired immunity.

Although we find a variety of studies on the SIR model [16]–[21], most of them are based on a mean-field approximation, in which one assumes independence of potentially

dependent random variables. One of the consequences of this approximation is the implicit introduction of terms [19] which can result in a large approximation error. The main goal of this paper is to introduce a theoretical framework for the analysis of the exact networked Markov process describing the SIR model, without resorting to mean-field approximations, and introduce an optimization framework to find an efficient allocation of resources to contain the expected number of accumulated infections over time. Our framework extends to a generalized SIR model where an infected node can be isolated by authorities (e.g., quarantine) for the purpose of suppressing an epidemic outbreak. In this case, we present an alternative optimization framework to distribute a finite amount of resources to suppress an epidemic outbreak.

This paper is organized as follows. After introducing mathematical preliminaries in Section II, we describe the networked SIR model (with and without isolated nodes), and state the resource allocation problems analyzed in this paper. In Sections III and IV, we introduce a convex optimization framework to provide solutions for these resource allocation problems. We illustrate the effectiveness of our results via numerical simulations in Section V.

A. Mathematical Preliminaries

An undirected graph is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, n\}$ is the set of nodes, and \mathcal{E} is the set of edges, consisting of distinct and unordered pairs $\{i, j\}$ for $i, j \in \mathcal{V}$. We say that i and j are adjacent if $\{i, j\} \in \mathcal{E}$. The adjacency matrix $A \in \mathbb{R}^{n \times n}$ of \mathcal{G} is defined as the $\{0, 1\}$ -matrix whose (i, j) entry is one if and only if i and j are adjacent.

For a positive integer n , define $[n] = \{1, \dots, n\}$. We let Id_n denote the identity matrix with dimension n and $O_{n,m}$ denote the $n \times m$ zero matrix. Let u_i denote the i th canonical basis in \mathbb{R}^p and define $U_{ij} = u_i u_j^\top$. By $\mathbb{1}_p$ we denote the p -vector whose entries are all one. A real matrix A , or a vector as its special case, is said to be nonnegative (positive), denoted by $A \geq 0$ ($A > 0$, respectively), if A is nonnegative (positive, respectively) entry-wise. We write $A \geq B$ if $A - B \geq 0$. The notations $A > B$, $A \leq B$, and $A < B$ are then defined in the obvious way. We denote the Kronecker product of matrices A and B by $A \otimes B$. We say that a square matrix is Hurwitz stable if all the eigenvalues of the matrix have negative real parts. Also, we say that a square matrix is Metzler if its off-diagonal entries are all non-negative.

For the proof of the main results of this paper, we need the following lemma on Metzler matrices:

Lemma 1.1 ([23, Lemma 1]): For a real number λ , a Metzler matrix $F \in \mathbb{R}^{n \times n}$ and nonnegative matrices $G \in \mathbb{R}^{n \times s}$

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Fig. 1. SIR model with infection rates β_i and recovery rates δ_i

and $H \in \mathbb{R}^{r \times n}$, the following statements are equivalent:

- F is Hurwitz stable and $-\mathbb{1}_r^\top H F^{-1} G < \lambda \mathbb{1}_s^\top$;
- There exists a positive vector $v \in \mathbb{R}^n$ satisfying the inequalities $v^\top F + \mathbb{1}_r^\top H < 0$ and $v^\top G < \lambda \mathbb{1}_s^\top$.

Finally, we recall basic facts about a class of optimization problems called geometric programs [24]. Let x_1, \dots, x_m denote m real positive variables. We say that a real-valued function f of $x = (x_1, \dots, x_m)$ is a *monomial function* if there exist $c > 0$ and $a_1, \dots, a_m \in \mathbb{R}$ such that $f(x) = c x_1^{a_1} \dots x_m^{a_m}$. Also, we say that f is a *posynomial function* if it is a sum of monomial functions of x . Given posynomial functions f_0, \dots, f_p and monomial functions g_1, \dots, g_q , the optimization problem

$$\begin{aligned} & \underset{x}{\text{minimize}} \quad f_0(x) \\ & \text{subject to} \quad f_i(x) \leq 1, \quad i = 1, \dots, p, \\ & \quad \quad \quad g_j(x) = 1, \quad j = 1, \dots, q, \end{aligned} \quad (1)$$

is called a *geometric program*. It is known [24] that a geometric program can be converted into a convex optimization problem. We call the constraints in (1) as *posynomial constraints*.

II. SIR MODEL OVER COMPLEX NETWORKS

In this section, we first give a brief overview of the networked SIR (susceptible-infected-removed) model (see, e.g., [1]). We also present an extended version of the networked SIR model, where the isolation of infected nodes is taken into account. We also state two resource allocation problems under study.

A. SIR model

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an undirected graph of n nodes whose adjacency matrix equals A . In the networked SIR model, each node in the graph can be in one out of three states: *susceptible*, *infected*, or *removed*. For convenience of notation, we represent the state of node i at time t by the $\{0, 1\}$ -variables $S_i(t)$, $I_i(t)$, and $R_i(t)$ that take value one if and only if node i is susceptible, infected, or removed, respectively. Then, the dynamics of the state of the nodes is described by the following transition probabilities

$$\Pr(I_i(t+h) = 1 \mid S_i(t) = 1) = \beta_i \sum_{j=1}^n a_{ij} I_j(t) h + o(h), \quad (2)$$

$$\Pr(R_i(t+h) = 1 \mid I_i(t) = 1) = \delta_i h + o(h), \quad (3)$$

where $t \geq 0$ and $h > 0$ are arbitrary. The constants $\beta_i > 0$ and $\delta_i > 0$ are called the *infection* and *recovery* rate of node i , respectively (see Fig. 1 for a schematic picture). The probability (2) indicates that a node i receives an infection from each of its infected neighbors with the instantaneous rate of β_i . Also, from the latter probability (3), the time it

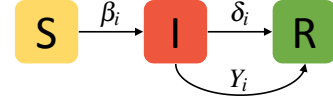


Fig. 2. SIR model with infection rates β_i , recovery rates δ_i , and recovery times Y_i

takes for node i to recover from an infection event follows the exponential distribution of mean $1/\delta_i$. We assume that a node is either susceptible or infected at time $t = 0$. We remark that, unlike in the SIS model widely studied in the literature, we do not allow a transition from the infected or removed states to the susceptible state, modeling the acquisition of immunity by nodes to spreading process.

Let $\sigma_S(t)$, $\sigma_I(t)$, and $\sigma_R(t)$ denote the number of susceptible, infected, and removed nodes at time t , respectively. In this paper, we measure the prevalence of the spreading process by the quantity

$$\lambda = \lim_{t \rightarrow \infty} E[\sigma_R(t)] - \sigma_I(0),$$

which equals the expected number of infections occurring after time $t = 0$ because an infected node will be eventually removed with probability one.

We assume that, for each node i , we can distribute a preventative resource to alter the value of β_i within a given interval $[\beta_i, \bar{\beta}_i] \subset (0, \infty)$ by paying a cost $f_i(\beta_i)$. Similarly we assume that we can distribute a corrective resource for tuning δ_i within another given interval $[\delta_i, \bar{\delta}_i] \subset (0, \infty)$ with an associated cost $g_i(\delta_i)$. Therefore, the total cost for achieving the specific infection rates $\{\beta_1, \dots, \beta_n\}$ and the recovery rates $\{\delta_1, \dots, \delta_n\}$ equals $\sum_{i=1}^n (f_i(\beta_i) + g_i(\delta_i))$. It is also assumed that we know which nodes in the graph are infected at the initial time $t = 0$.

Now we can state the first problem studied in this paper:

Problem 2.1 (Resource allocation): Given a set of initially infected nodes, an available budget $\bar{C} > 0$, and a desired control level $\bar{\lambda} > 0$, find $\beta_i \in [\beta_i, \bar{\beta}_i]$ and $\delta_i \in [\delta_i, \bar{\delta}_i]$ ($i \in [n]$) such that $\lambda \leq \bar{\lambda}$ and

$$\sum_{i=1}^n (f_i(\beta_i) + g_i(\delta_i)) \leq \bar{C}. \quad (4)$$

B. SIR Model with Isolation

The above problem statement is not practical when there are no medical resource able to tune the recovery rate of infected patients. An alternative action in this situation is to use social distancing, such as quarantine [25]. A typical question in this context is the following: *How fast should we isolate infected nodes to effectively prevent an epidemic outbreak?*

In what follows, we introduce an extended SIR model that incorporates the isolation of infected nodes. We assume that, once a node i becomes infected, the node is removed (quarantined) according to a stochastic process, which is independent of the natural recovery process (as depicted in Fig. 2). Let X_i denote the random variable following the

exponential distribution with mean $1/\delta$ and Y_i the time it takes to remove the node. Then, the overall length of time from the infection of i to its removal (either by natural recovery or manual removal) is given by

$$Z_i = \min(X_i, Y_i). \quad (5)$$

We assume that the distribution of the random variable Y_i is parametrized by positive variables $\gamma_i = (\gamma_{i1}, \dots, \gamma_{iq_i})$, and that we can tune the distribution by paying a cost $h_i(\gamma_i)$. We also assume that, as in Problem 2.1, we can tune the infection rates β_i with a cost $g_i(\beta_i)$. We can then formulate the second problem studied in this paper:

Problem 2.2 (Resource allocation with isolation): Given a set of initially infected nodes, an available budget $\bar{C} > 0$, and a desired control level $\bar{\lambda} > 0$, find $\beta_i \in [\underline{\beta}_i, \bar{\beta}_i]$ and $\gamma_i \in \prod_{k=1}^{q_i} [\underline{\gamma}_{ik}, \bar{\gamma}_{ik}]$ ($i = 1, \dots, n$) such that $\lambda \leq \bar{\lambda}$ and

$$\sum_{i=1}^n (g_i(\beta_i) + h_i(\gamma_i)) \leq \bar{C}. \quad (6)$$

III. RESOURCE ALLOCATION WITHOUT ISOLATION

This section gives a solution to Problem 2.1. We specifically show that Problem 2.1 is feasible if a set of posynomial constraints is feasible. For this purpose, we present an n -states linear model that upper-bounds the averaged dynamics of the *exact* SIR model. We notice that, since the SIR model is a Markov process having 3^n states, its direct analysis is computationally hard, if not possible, even when the size n of the graph is small.

We start with describing the SIR model by stochastic differential equations. Let us denote by N_μ a Poisson counter of rate μ . We assume that all the Poisson counters appearing in the paper are stochastically independent. Then, from (2) and (3), the evolution of the nodal states can be exactly described by the stochastic differential equations:

$$\begin{aligned} dS_i &= -S_i \sum_{j=1}^n a_{ij} I_j dN_{\beta_i}, \\ dI_i &= -I_i dN_{\delta_i} + S_i \sum_{j=1}^n a_{ij} I_j dN_{\beta_i}, \\ dR_i &= I_i dN_{\delta_i}, \end{aligned} \quad (7)$$

for $i = 1, \dots, n$. Using these representations, we can prove the following proposition:

Proposition 3.1: Define the $n \times n$ diagonal matrices J , B , and D by

$$J_{ii} = S_i(0), \quad B_{ii} = \beta_i, \quad D_{ii} = \delta_i$$

for each $i \in [n]$. Let $\bar{\lambda} > 0$ be given. Then, we have $\lambda < \bar{\lambda}$ if there exists a positive $v \in \mathbb{R}^n$ satisfying the following inequalities:

$$v^\top JBA + \mathbb{1}_n^\top D < v^\top D, \quad (8a)$$

$$v^\top I(0) < \bar{\lambda} + \sigma_I(0). \quad (8b)$$

Proof: From (7) it follows [18] that

$$\frac{d}{dt} E[I_i(t)] = -\delta_i E[I_i(t)] + \beta_i \sum_{j=1}^n a_{ij} E[I_j(t) S_i(t)], \quad (9)$$

and $(d/dt)E[R_i(t)] = \delta_i E[I_i(t)]$. From the latter equation, the $\{0, 1\}^n$ -valued stochastic processes $R = [R_1 \dots R_n]^\top$ and $I = [I_1 \dots I_n]^\top$ satisfy $(d/dt)E[R(t)] = DE[I(t)]$. Therefore,

$$\lim_{t \rightarrow \infty} E[R(t)] = D \int_0^\infty E[I(t)] dt. \quad (10)$$

Let us evaluate the integral $\int_0^\infty E[I(t)] dt$. Since S_i is non-increasing as a function of t , we have $S_i(t) \leq S_i(0) = J_{ii}$. Therefore, by (9) we have

$$\frac{d}{dt} E[I_i(t)] \leq -\delta_i E[I_i(t)] + J_{ii} \beta_i \sum_{j=1}^n a_{ij} E[I_j(t)]$$

for every i . Therefore, we obtain

$$\frac{d}{dt} E[I(t)] \leq (JBA - D)E[I(t)]$$

and, hence,

$$E[I(t)] \leq \exp((JBA - D)t) I(0) \quad (11)$$

by the comparison principle [26]. Since $JBA - D$ is Hurwitz stable by (8a) and Lemma 1.1, we can integrate (11) to obtain

$$\begin{aligned} \int_0^\infty E[I(t)] dt &\leq \int_0^\infty \exp((JBA - D)t) I(0) dt \\ &= -(JBA - D)^{-1} I(0). \end{aligned}$$

From this inequality and (10), we see that

$$\begin{aligned} \lambda &= \mathbb{1}_n^\top \lim_{t \rightarrow \infty} E[R(t)] - \sigma_I(0) \\ &\leq -\mathbb{1}_n^\top D (JBA - D)^{-1} I(0) - \sigma_I(0). \end{aligned}$$

Therefore, by Lemma 1.1, we can conclude $\lambda \leq \bar{\lambda}$ from the existence of a positive vector $v \in \mathbb{R}^n$ satisfying (8). This completes the proof of the proposition. ■

The inequalities in (8) allow us to achieve an efficient resource distribution via checking the feasibility of posynomial constraints, under the assumption that the cost functions f_i and g_i are posynomials for every $i \in [n]$:

Theorem 3.2: Assume that f_i and g_i are posynomials for every $i \in [n]$. Then, $\{\beta_i\}_{i=1}^n$ and $\{\delta_i\}_{i=1}^n$ solve Problem 2.1 if there exists a positive $v \in \mathbb{R}^n$ satisfying the following posynomial constraints:

$$(4), (8), \quad \underline{\beta}_i \leq \beta_i \leq \bar{\beta}_i, \quad \text{and} \quad \underline{\delta}_i \leq \delta_i \leq \bar{\delta}_i. \quad (12)$$

Proof: It is easy to check that all the constraints in (12) are posynomial constraints, provided f_i and g_i are posynomials for every i . Moreover, from Proposition 3.1, we can easily see that the rates β_i and δ_i satisfying the constraints (12) solve Problem 2.1. ■

IV. RESOURCE ALLOCATION WITH ISOLATION

This section presents a solution to Problem 2.2. As in the previous section, we show that Problem 2.2 is feasible if a set of posynomial constraints is feasible, under the general assumption that the removal times Y_i can be described by a general class of distributions called phase-type distributions [27].

We start our presentation by reviewing phase-type distributions [27]. Consider a time-homogeneous Markov process

in continuous-time with $p+1$ ($p \geq 1$) states such that states $1, \dots, p$ are transient and state $p+1$ is absorbing. The infinitesimal generator of the process, that is, the matrix of transition rates, is then necessarily of the form

$$\begin{bmatrix} \Pi & w \\ 0 & 0 \end{bmatrix}, \quad w = -\Pi \mathbb{1}_p,$$

where $\Pi \in \mathbb{R}^{p \times p}$ is an invertible Metzler matrix with non-positive row-sums. Let $\begin{bmatrix} \phi \\ 0 \end{bmatrix} \in \mathbb{R}^{p+1}$ ($\phi \in \mathbb{R}^p$) denote the initial distribution of the Markov process. Then, the time to absorption into the state $p+1$, denoted by (ϕ, Π) , is called a *phase-type distribution*. It is known that the set of phase-type distributions is dense in the set of positive valued distributions [27], even when $\phi = u_1$. Moreover, there exists an efficient fitting algorithm to approximate a given arbitrary distribution by a phase-type distribution [27].

We can now state our assumption on the distribution of the removal times Y_i :

Assumption 4.1: There exists $p \geq 0$ such that, for each $i \in [n]$, Y_i follows the phase-type distribution (u_1, Π_i) , with $\Pi_i \in \mathbb{R}^{p \times p}$ being parametrized by positive variables $\gamma_i = (\gamma_{i1}, \dots, \gamma_{ip})$.

The rest of this section is devoted to giving a solution to Problem 2.2. We start with showing that the overall removal time Z_i given in (5) follows a phase-type distribution:

Lemma 4.2: Define $\Pi'_i = \Pi_i - \delta_i \text{Id}_p$. Then Z_i follows the phase-type distribution (u_1, Π'_i) .

Proof: We first recall that the cumulative distribution function of the phase-type distribution (ϕ, Π) equals $F(t) = 1 - \phi \exp(t\Pi) \mathbb{1}_p$. We also recall that, given independent random variables X_1 and X_2 having the cumulative distribution functions F_1 and F_2 , the cumulative distribution function of the random variable $\min(X_1, X_2)$ equals $1 - (1 - F_1(t))(1 - F_2(t))$. From the above facts, the cumulative distribution function of Z_i equals

$$\begin{aligned} F(t) &= 1 - (1 - (1 - e^{-\delta_i t})) (1 - (1 - u_1 \exp(t\Pi_i) \mathbb{1}_p)) \\ &= 1 - e^{-\delta_i t} u_1 \exp(t\Pi_i) \mathbb{1}_p \\ &= 1 - u_1 \exp(t(\Pi_i - \delta_i \text{Id}_p)) \mathbb{1}_p, \end{aligned}$$

which coincides with the cumulative distribution function of (u_1, Π'_i) , as desired. ■

From Lemma 4.2 we see that, in our SIR model with isolations, the infection rates are constants and the overall removal times Z_i follow phase-type distributions. Therefore, this SIR model has a similar structure with the SIS model studied in [22], where it is assumed that the infection rates are constants while the recovery (i.e., the transition from the infected state to the susceptible state) occurs following phase-type distributions. Hence, following the same argument as in [22], we can describe our SIR model with isolations using stochastic differential equations:

Proposition 4.3: Define $w'_i = [w'_{i1} \dots w'_{ip}]^\top = -\Pi'_i \mathbb{1}_p$. Let the $\{0, 1\}$ -valued stochastic processes S_i, R_i and the $\{0, 1\}$ -valued stochastic processes $\tilde{I}_i = [\tilde{I}_{i,1} \dots \tilde{I}_{i,p}]^\top$

($i = 1, \dots, n$) follow the stochastic differential equations:

$$\begin{aligned} dS_i &= -S_i \sum_{j=1}^n a_{ij} \mathbb{1}_p^\top \tilde{I}_j dN_{\beta_i}, \\ d\tilde{I}_i &= \sum_{\ell,m=1}^p (U_{m\ell} - U_{\ell\ell}) \tilde{I}_i dN_{\Pi'_{i,\ell m}} - \sum_{\ell=1}^p U_{\ell\ell} \tilde{I}_\ell dN_{w'_{i\ell}} \\ &\quad + u_1 S_i \sum_{j=1}^n a_{ij} \mathbb{1}_p^\top \tilde{I}_j dN_{\beta_i}, \end{aligned} \quad (13)$$

$$dR_i = \sum_{\ell=1}^p \tilde{I}_{i,\ell} dN_{w'_{i\ell}}, \quad (14)$$

with the initial conditions

$$\begin{aligned} (S_i(0), \tilde{I}_i(0), R_i(0)) &= \begin{cases} (0, u_1, 0), & \text{if } i \text{ is infected at time } 0, \\ (1, 0, 0), & \text{otherwise.} \end{cases} \end{aligned}$$

Define

$$I_i(t) = \mathbb{1}_p \tilde{I}_i(t).$$

Then, in the SIR model with isolations, a node i is susceptible, infected, or removed at time t if and only if $S_i(t) = 1$, $I_i(t) = 1$, or $R_i(t) = 1$, respectively.

Proof: We refer the readers to the proof of [22, Proposition 3.1]. The details are omitted. ■

Based on Proposition 4.3, we can prove the following proposition:

Proposition 4.4: Let $\tilde{\lambda} > 0$ be given. Then, we have $\lambda < \tilde{\lambda}$ if there exists a positive vector $v \in \mathbb{R}^{np}$ satisfying the following inequalities:

$$v^\top \left(\bigoplus_{i=1}^n (\Pi'_i)^\top + (JBA) \otimes (u_1 \mathbb{1}_p^\top) \right) + \mathbb{1}_n^\top \bigoplus_{i=1}^n (w'_i)^\top < 0, \quad (15)$$

$$v^\top \tilde{I}(0) < \tilde{\lambda} + \sigma_f(0). \quad (16)$$

Proof: Define $\tilde{I} = [\tilde{I}_1^\top \dots \tilde{I}_n^\top]^\top$. From (14) it follows that

$$\begin{aligned} (d/dt)E[R_i(t)] &= \sum_{\ell=1}^p E[\tilde{I}_{i,\ell}(t)] w'_{i\ell} \\ &= (w'_i)^\top E[\tilde{I}_i(t)] \end{aligned}$$

and therefore

$$(d/dt)E[R(t)] = \left(\bigoplus_{i=1}^n (w'_i)^\top \right) E[\tilde{I}(t)].$$

This equation shows that

$$\lim_{t \rightarrow \infty} E[R(t)] = \left(\bigoplus_{i=1}^n (w'_i)^\top \right) \int_0^\infty E[\tilde{I}(t)] dt. \quad (17)$$

On the other hand, in the same way as in [22], from (13) we observe $(d/dt)E[\tilde{I}(t)] \leq \mathcal{A}E[\tilde{I}(t)]$, where

$$\mathcal{A} = \bigoplus_{i=1}^n (\Pi'_i)^\top + (JBA) \otimes (u_1 \mathbb{1}_p^\top).$$

Therefore,

$$E[\tilde{I}(t)] \leq \exp(\mathcal{A}t) \tilde{I}(0). \quad (18)$$

Notice that, by (15) and Lemma 1.1, the matrix \mathcal{A} is Hurwitz stable. Therefore, in the same way as in the proof of Proposition 3.1, from (17) and (18) it follows that the condition stated in the theorem is sufficient for $\lambda \leq \bar{\lambda}$. The details are omitted. ■

Based on Proposition 4.4, we can derive the following theorem, which enables us to solve Problem 2.2 by checking the feasibility of posynomial constraints:

Theorem 4.5: Let $\mathcal{D}\Pi_i$ and $\mathcal{O}\Pi_i$ denote the diagonal and off-diagonal parts of Π_i , respectively. For all $i \in \{1, \dots, n\}$, assume that

- 1) the entries of w_i and $\mathcal{O}\Pi_i$ are posynomials in γ_i ;
- 2) the entries of $-\mathcal{D}\Pi_i$ are monomials in γ_i ;
- 3) the cost functions g_i and h_i are posynomials in β_i and γ_i , respectively.

Moreover, for all $i \in [n]$ and $\ell \in [p]$, let $\kappa_{i\ell}$ and $\alpha_{i\ell}$ be positive constants such that

$$\kappa_{i\ell}(-\Pi_{i,\ell\ell})^{\alpha_{i\ell}} \leq (-\Pi_{i,\ell\ell}) + \delta_i \quad (19)$$

for all possible values of γ_i . Then, the parameters $\beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_n$ solve Problem 2.2 if there exists a positive vector $v \in \mathbb{R}^{np}$ satisfying the posynomial constraints:

$$\begin{aligned} v^\top \left(\bigoplus_{i=1}^n (\mathcal{O}\Pi_i)^\top + (JBA) \otimes (u_1 \mathbb{1}_p^\top) \right) \\ + \mathbb{1}_p^\top \bigoplus_{i=1}^n w_i^\top + \mathbb{1}_n^\top (D \otimes \mathbb{1}_p) < v^\top \bigoplus_{i=1}^n \bigoplus_{\ell=1}^p \kappa_{i\ell} (-\Pi_{i,\ell\ell})^{\alpha_{i\ell}}, \end{aligned} \quad (20a)$$

$$(6), (16), \underline{\beta}_i \leq \beta_i \leq \bar{\beta}_1, \gamma_i \in \prod_{k=1}^{q_i} [\underline{\gamma}_{ik}, \bar{\gamma}_{ik}]. \quad (20b)$$

Proof: It is straightforward to see that the conditions in (20) are all posynomial constraints with respect to variables v, β_i , and γ_i ($i \in [n]$) under the assumptions stated in the theorem. Assume that $\beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_n$ satisfy the constraints (20a) and (20b). Then, by the conditions on the constants κ_ℓ and α_ℓ , we have

$$\bigoplus_{i=1}^n \bigoplus_{\ell=1}^p \kappa_{i\ell} (-\Pi_{i,\ell\ell})^{\alpha_{i\ell}} \leq \delta \text{Id}_{np} + \bigoplus_{i=1}^n (-\mathcal{D}\Pi_i).$$

Substituting this inequality to (20a) indeed yields (15). This observation and the constraints (20b) guarantee $\lambda \leq \bar{\lambda}$ by Proposition 4.4. ■

Remark 4.6: We can use a bisection search to find the pair $(\kappa_{i\ell}, \alpha_{i\ell})$ that satisfy (19) and, moreover, minimize the maximum difference between the left- and right-hand sides of (19). The details are omitted due to limitations of space.

V. NUMERICAL EXAMPLES

We present numerical examples in this section, for both the cases without and with isolations (i.e., quarantine). We let \mathcal{G} be the graph of a part of a social network of $n = 68$ nodes. The adjacency matrix of the graph has the spectral radius $\rho = 10.61$. Also, we randomly choose and fix four initially infected nodes from the graph.

We first consider the case without isolations. For each $i \in [n]$, let $\underline{\beta}_i = \underline{\beta} = 0.00266$, $\bar{\beta}_i = \bar{\beta} = 0.0133$, $\underline{\delta}_i = \underline{\delta} = 0.05$, and

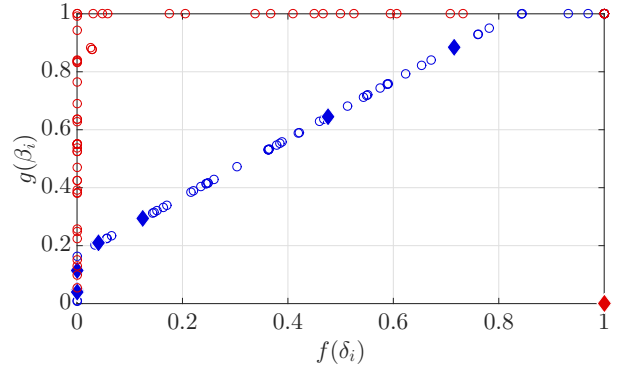


Fig. 3. Correction versus prevention per node. Red: Proposed. Blue: The optimal allocation for the SIS model [9]. Diamond markers correspond to the initially infected nodes.

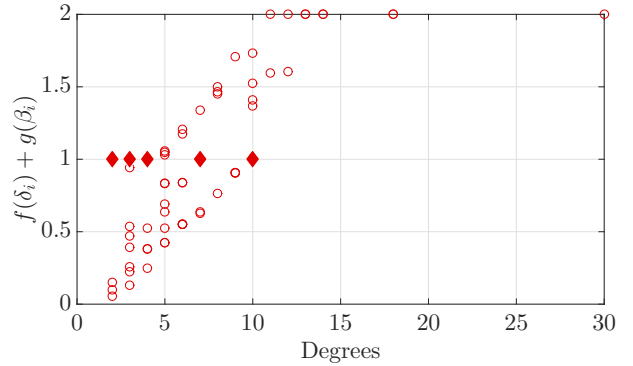


Fig. 4. Sub-optimal investments per node versus degrees. Diamond markers correspond to the initially infected nodes

$\bar{\delta}_i = \bar{\delta} = 0.1$. We use the cost functions $f_i(\beta_i) = c_1 \beta_i^{-1} + c_2$ and $g_i(\delta_i) = c_3 \delta_i + c_4$, where the real constants $c_1 > 0$, c_2 , $c_3 > 0$, and c_4 are chosen in such a way that $f_i(\underline{\delta}) = 1$, $f_i(\bar{\delta}) = 0$, $g_i(\underline{\beta}) = 0$, and $g_i(\bar{\beta}) = 0$. We let $\bar{C} = n = 68$. To find sub-optimal resource allocations, we minimize $\bar{\lambda}$ subject to the constraints in (12). This optimization problem is a geometric program because $\bar{\lambda}$ is trivially a monomial. The scatter plot of the costs from the obtained recovery rates and infection rates is shown in Fig. 3 (red plots). We can observe an interesting difference of the obtained cost allocation from the one based on the SIS model [9] (blue plots). In Fig. 4, we show the scatter plot of the sub-optimal investments on each node versus degrees of the nodes. Using Monte Carlo simulation, we find that the proposed allocation achieves $\lambda = 2.57$, which is 40% less than $\lambda = 4.38$ obtained from the allocation [9] optimized for the SIS model.

We then consider the case with isolations. We model the isolation times Y_i by Erlang distributions with mean $\gamma_i > 0$ and shape p . We let γ_i be the design variable. Since an Erlang distribution is a p -sum of independent and identically distributed exponential distributions, Y_i approximates a normal distribution when p is large. Moreover, Y_i is the phase type

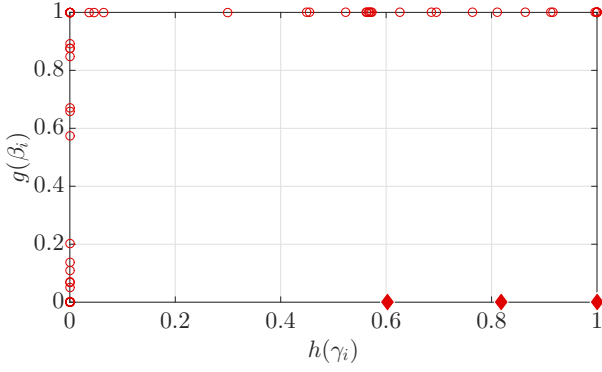


Fig. 5. Isolation versus prevention per node. Diamond markers correspond to the initially infected nodes.

distribution (e_1, Π_i) with

$$\Pi_i = \begin{bmatrix} -p/\gamma_i & p/\gamma_i & & O \\ & \ddots & \ddots & \\ & & \ddots & p/\gamma_i \\ O & & & -p/\gamma_i \end{bmatrix} \in \mathbb{R}^{p \times p}.$$

Since

$$w_i = \begin{bmatrix} O_{p-1,1} \\ p/\gamma_i \end{bmatrix}, \quad O\Pi_i = \begin{bmatrix} O_{p-1,1} & (p/\gamma_i)\text{Id}_{p-1} \\ 0 & O_{1,p-1} \end{bmatrix},$$

and $-\mathcal{D}\Pi_i = (p/\gamma_i)\text{Id}_p$, all the assumptions in Theorem 3.2 are satisfied. We choose the cost function for γ_i as $h_i(\gamma_i) = c_5/\gamma_i + c_6$, where $c_5 > 0$ and c_6 are constants such that $h(\gamma_i) = 1$ and $h(\gamma_i) = 0$. This choice is based on an assumption that we have to pay the more cost to achieve the faster response to patients. We fix $\delta_i = 0.1$ and $\bar{C} = n$. To find sub-optimal resource allocations, we minimize $\bar{\lambda}$ subject to the constraints in (20). We show the scatter plot of the sub-optimal resource allocation in Fig. 5, where we can observe a similar pattern as Fig. 3.

VI. CONCLUSION

In this paper, we have proposed a convex optimization framework to contain an epidemic outbreak in the networked SIR models. We have developed a framework to find a sub-optimal resource allocation to contain the accumulated number of infections over time. We have then extended our results to a networked SIR model allowing isolations (quarantines), where infected nodes can be removed from the population. We have then illustrated the efficiency of our framework via numerical simulations in a real social network.

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